{sample section from the book: "The Afterlife and the True Nature of Reality" by John T. Mennella}

{this is a partial chapter}

{*One section in my book – specifically in* Chapter 14: Cracks in the Foundation of Our Understanding of Reality – *utilizes mathematical notation. Since some Kindle devices have difficulty displaying mathematical symbols, I am offering those portions from Section 14.2 here for viewing and/or downloading.}*

{*From Section 14.2*}

Imaginary Numbers

Numbers represent the absolute bedrock of mathematics. It is easy to forget that the complex and abstract discipline of modern mathematics sprang from something as basic and obvious as simple numbers. Whether it was the need of an ancient shepherd to tally the sheep in his herd, or the earliest desire to measure time or distance, the field of mathematics – this exotic, alien world of symbols and formulae and ultra-logic – can be traced back to the humble roots of number. Before I can get to the point of this section – imaginary numbers – I must first provide some background on how numbers are categorized.

Numbers evolved over time, beginning inevitably with the simple and practical – the counting numbers: 1, 2, 3, etc. – and expanding into ever more conceptual representations – fractions, decimals, and so on. These different ways of representing numerical quantities, or different classifications of numbers, are today expressed in the language of sets. Set theory was first conceived by mathematician Georg Cantor. In mathematical parlance a set is defined as a "well-defined collection of distinct objects" where the objects can be anything – numbers, things, ... even other sets.[i] For our purposes here we will limit our consideration to sets of numbers. (note: sets are typically denoted by squiggly brackets {}, and the objects making up a set are referred to as the *elements* of the set).

While you may never have had cause to think of numbers within the framework of set theory, the truth is that you will find the underlying notions and definitions familiar and comprehendible from your everyday use and understanding of commonplace numbers. Number sets, as with numbers themselves, begin at the most basic level and then gradually grow into ever more elaborate representations. The most fundamental set of numbers is known as the **Natural numbers** (also known, for an immediately obvious reason, as the Counting numbers): {1,2,3,...}. This set is clearly a set containing an infinite number of elements, and such will be the case for each number set we will discuss. If we add the element zero to the set of Natural numbers we create a new number set called the set of **Whole numbers**: {0,1,2,3,...}. So far, so good; nothing weird or esoteric here, just our everyday numbers being grouped together in different ways. As you get the hang of this, you are probably thinking ahead and anticipating the inclusion of fractions to create yet another number set – and you are correct; but before we can open that can of worms we must first consider one more set of "whole" (in the sense of non-fractional) numbers. Probably around the eighth grade or so, you were introduced to a mathematical concept which, at that point in your life, may have seemed a bit bizarre and unreal: the idea of "negative numbers." You learned that each number with which you were familiar (excepting zero) had a negative counterpart: 2 and -2, 7 and -7, etc. (If your experience was anything like my own, you inevitably referred to such numbers verbally by saying "*minus* 2" or "*minus* 7" and you were immediately rebuked by your math teacher who insisted that you use the word "negative" in lieu of "minus."). When we adjoin the negative counterparts to the set of Whole numbers we have yet another number set known as the set of **Integers**: {... -3,-2,-1,0,1,2,3,...}.

Needless to say, as humanity became comfortable with the idea of using numbers to represent quantity and measurement, at some point the need arose to represent portions or parts of whole quantities, and this need gave rise to the idea of fractions. If we add fractions to the set of integers we form the new number set known as the set of **Rational numbers**. Two important points need be noted here: first, we are using the term *fraction* here in the strict sense of the ratio of two integers, numerator over denominator (where the denominator cannot be zero); and second, it is difficult to represent the set of Rational numbers using the convenient symbolic set notation we have heretofore been using (i.e., the {} brackets), the reasons being that there are *so many* fractions possible and it is exceedingly difficult to attempt to present a select sample in strict numerical sequence. In other words, for integers and whole numbers we know that 2 follows directly after 1 and 3 follows directly after 2, and so on; but what follows *directly after*, say, ½ ? Choose any fraction that comes after ½ and you can still think of another fraction that comes between that fraction and ½. There *is* a way to represent the Rational numbers using set notation, but it's a bit complicated and not germane to this discussion so I won't go into it. Suffice it to say that it does not employ the "representative sample" technique used to typify the Natural numbers, Whole numbers, and Integers. For the sake of convenience, mathematicians also use capitol letters to represent the various number sets, so, for example, **N** represents the set of Natural numbers, **Z** the integers, and **Q** the Rational numbers.

Continuing on with our development of number sets: as you know, there is a way to represent parts of whole quantities other than via fractions – namely, decimal numbers. While it is true that every fraction can be represented in decimal notation (e.g., ½ = .5), the converse is not true: there are many decimal numbers that cannot be represented as fractions; such numbers – decimal numbers that cannot be expressed as the ratio of two integers – are called *irrational numbers* (as distinct from *rational numbers* which, as we recently saw, are numbers that *can be* represented as the ratio of two integers). Examples of irrational numbers would be pi (π) – which is often *approximated* as 3.14 – and the square root of 2 ($\sqrt{2}$). When we group the irrational numbers together with the set of Rational numbers, we form a new set known as the set of **Real numbers** (**R**), often referred to informally as "the Reals." For the purposes of everyday life, the set of Real numbers constitutes the extent of our experience with numbers – the Real numbers are all the numbers most of us will ever need or see. But that doesn't mean the Real numbers are the end of the story, and this is where things, number-wise, start to get weird.

It turns out that mathematicians have conceived of another species of number that lies outside the realm of the Real numbers. This point alone is worth reflecting on: If the Real numbers constitute all of the numbers we use and consider in everyday life – namely, the Integers (positive and negative whole numbers and zero), the rational numbers (fractions), and the irrational numbers (decimal numbers that cannot be represented as fractions) – then what else is there? The set of Real numbers seems to account for every possible number one can imagine – try to think of a number that is *not* a positive or negative whole number, *not* zero, *not* a fraction, and *not* a decimal. There aren't any, at least not within the conceptual framework of the notion of "number" as we understand it. That's why the set of Real numbers is called "Real" – they are the numbers that we need and use to function in our *real*ity, to measure and quantify our reality. So, what other kind of number could there possibly be? Well, if it's not a Real number, then it must be ... *unreal*. And, by God, it is! This new species of number is called an *imaginary number*, and to create it requires breaking some rules.

Before we can understand what an imaginary number is, we must first review the mathematical notion of square root. The square root of a number ***k*** is a number which when multiplied by itself gives ***k*** as the result. So, for example, the square root of 4 is 2, since 2 x 2 = 4; likewise, the square root of 25 is 5, since 5 x 5 = 25. Simple enough. Except... we mustn't forget to consider the negative numbers. Recall from your school days that, when you first began working with integers (sometimes referred to as "signed numbers"), you learned some important rules for multiplying these signed numbers, namely:

- a positive times a positive yields a positive result: 2 x 2 = 4
- a negative times a negative yields a positive result: (-2) x (-2) = 4
- a positive times a negative yields a negative result: 2 x (-2) = -4
- a negative times a positive yields a negative result: (-2) x 2 = -4

In mathematics parlance, we refer to the result of a multiplication operation as a *product*. Thus we can restate the above rules more succinctly as follows:

- the product of two numbers with the same sign is positive.
- the product of two numbers with opposite signs is negative.

We now see that the number 4 actually has two square roots, namely 2 and -2. More generally, every positive number has two square roots, one being positive and the other being negative (so, for example, the square roots of 25 are 5 and -5). Zero – which is considered to be neither positive nor negative – has only one square root, namely itself (zero). But what about square roots of negative numbers? This is where things get interesting.

The rules for multiplying signed numbers, considered together with the concept of square root, lead to an important realization: Since the product of two numbers with the same sign must always be positive, and since the square root of a number is the product of a *number times itself*, we must conclude that *a negative number cannot have a square root*. Think about it: a "number times itself" will always be a case of a positive times a positive or a negative times a negative, and will thus always yield a positive result. In other words, there is no possible way we can have a situation wherein a number times itself will give a negative result. Let's look at an example: Suppose we are asked to calculate the square root of -4 ($\sqrt{-4}$); this is asking us to find a number which when multiplied by itself will yield -4 as the result. Not possible! *There is no number which, when multiplied by itself, will give us -4 as the result*: 2 x 2 = 4 (since a positive times a positive gives a positive result), and (-2) x (-2) = 4 (since a negative times a negative gives a positive result). In other words, a number times itself – regardless of whether that number is positive or negative – will always yield a *positive result*. Thus a negative number cannot have a square root.

Unless...

Remember my earlier observation that mathematics is the "exact science" that is not exactly exact? If you're willing to break the rules, then anything is possible. We can resolve this annoying inconvenience by *defining a new number into existence*. "Wait a minute, that's cheating!" you say? Remember my math professor "explaining" to me that 1^{∞} is undefined because it is *defined* that way? Apparently the act of "defining" things in mathematics allows for the circumventing of fundamental principles. But such "thinking outside the box" often leads to fascinating results.

Mathematicians resolved the dilemma of the impossibility of square roots of negative numbers by expanding the concept of number to include a new type of number, aptly named (as previously noted) the *imaginary number*. The imaginary number (or imaginary unit) is denoted by the letter $\boldsymbol{i}$ and is defined as follows:

$$\boldsymbol{i} = \sqrt{-1}$$

And from this simple sleight-of-hand we have now allowed for the possibility, and very existence, of square roots of negative numbers, since:

$$\text{if } \boldsymbol{i} = \sqrt{-1} \quad \text{then it follows that} \quad \boldsymbol{i}^2 = -1$$

Then we can say that, for example:

$$\sqrt{-4} = 2\boldsymbol{i} \ \ (\text{since } 2\boldsymbol{i} \text{ x } 2\boldsymbol{i} = 4\boldsymbol{i}^2 = 4 \text{ x } (-1) = -4)$$

If that last part is confusing to you, don't sweat it. The important point here is that mathematicians have defined into existence a new species of number, the imaginary number $\boldsymbol{i}$, which they've defined as being equal to $\sqrt{-1}$.

I have to admit that when I first learned about imaginary numbers I found the notion troubling because it appeared to me that fundamental concepts of mathematics were being violated with casual recklessness. As I noted above, mathematics provides a clear definition of what it means to calculate a square root, and it also provides very clear rules for multiplying signed numbers: together, these concepts preclude the possibility of a square root of a negative number. Mathematics is a very strict and rigorous enterprise; to circumvent its principles and doctrines by defining forbidden entities into existence struck me as, at the very least, sloppy, and, at worst, a degrading violation of the foundational framework upon which the entire edifice of mathematics is constructed. Now, in hindsight, I suspect my attitude was naïve and narrow-minded, but I have to go on record as stating that, even to this day, I get a little twinge of uneasiness whenever I think about the concept of imaginary numbers. My mathematical neuroses aside, it turns out that imaginary numbers are an instrument of convenience enabling complex computations that would otherwise be incredibly difficult or impossible.

The truth is that imaginary numbers play important roles in mathematics and science, and they are by no means considered "imaginary" by mathematicians. Imaginary numbers serve as a useful mathematical tool in enabling the discovery of new theorems as well as in expanding the reach of existing theorems. Indeed, it was this utility of imaginary numbers that persuaded mathematicians, some of whom (such as Descartes) initially rejected the concept, that there must be a logic to this strange beast and it should be taken seriously.[ii] Furthermore, imaginary numbers have proven to be an invaluable aid to physicists and engineers in making complex calculations related to signal processing, fluid dynamics, electric current, relativity theory, quantum mechanics, and even in biology for analyzing neuronal activity in the brain. [iii] Also, imaginary numbers actually have a geometric interpretation (though it is not linear [iv] and involves rotations in a coordinate plane [v]). Thus, as a handy tool that facilitates difficult mathematical calculations and computations, imaginary numbers can be viewed as "an upgrade to our number system, just like zero, decimals and negatives were."[vi]

The truly amazing thing about imaginary numbers is that *they work*. And this, then, raises a critically important question – one that is highly relevant to our discussion of the nature of reality: How is it that this artificial – arguably illegitimate – mathematical construct turns out to be helpful in solving equations related to actual natural processes? Perhaps imaginary numbers actually do have a "true existence" on some level of reality outside the scope of our everyday experience. This possibility is not as far-fetched as it sounds, and is, in fact, strongly hinted at by the most elegant, beautiful and sublime equation in all of mathematics: Euler's Identity. Let's take a look at this most fascinating mathematical statement.

Euler's Identity

$$e^{i\pi} + 1 = 0$$

This equation combines five of the most important mathematical constants into one deceptively simple formula. We have:

e: Euler's number, the base of the natural logarithms.
π: pi, the ratio of the circumference of a circle to its diameter.
i: the imaginary unit.
1: the multiplicative identity.
0: the additive identity.
(also note that 0 and 1 are the two digits of the binary number system, the simplest number system possible and the mathematical basis of computer logic).

To have these five important constants (and *only* these constants) appear together in one simple, elegant equation is nothing short of astounding. The implications of this formula with regards to the interrelationship of these five values can scarcely be imagined: *What*, exactly, is this formula telling us? And how is it that the *imaginary* number ***i*** gets equal billing with the other four critically important *real number* constants? I, personally, cannot help but read into this equation the suggestion that ***i*** is more than an artificial construct created for the convenience of mathematicians; to me, it says that ***i*** has real meaning in the real world, on par with ***e*, π, 1** and **0**.

Also, note that by subtracting 1 from both sides of the equation, we arrive at the result:

$$e^{i\pi} = -1$$

Really? How does an expression as complicated as $e^{i\pi}$, a numerical expression that involves two non-terminating decimal numbers and an imaginary number, simplify down to plain old -1 ? Seriously, think about what this expression entails numerically:

e (Euler's number) is a non-terminating decimal equal to: 2.7182818284...
π (pi) is a non-terminating decimal equal to: 3.1415926535...
i (the imaginary unit) is equal to $\sqrt{-1}$

Thus $e^{i\pi}$ actually means $(2.7182818284...)^{(\sqrt{-1})(3.1415926535...)}$

And that ridiculously inexact and convoluted expression calculates to a nice, simple number like -1 ? Come on! But it does; that's the inexplicable truth of it.

At the risk of sounding over-dramatic, I honestly sense that there is an important and profound piece of information hidden in Euler's equation, something that goes well beyond mathematics into realms philosophical, alchemical, and eerily magical.

Another mathematical expression that I find to be as eye-poppingly astounding as Euler's Identity is the following:

$$i^{i} = \frac{1}{\sqrt{e^{\pi}}}$$

Let's take a moment to fully comprehend what this amazing equation is saying. On the left-side we have the imaginary number ***i*** raised to the ***i***th power – in other words, we have an imaginary number raised to an imaginary exponent (I can't even *begin* to guess what that means, or how to interpret it) hence the left-side involves only imaginary quantities. On the right-side of the equation we have an expression involving **1**, ***e***, and ***π***; these are all numerical quantities that are Real numbers. So what we have here is *a mathematical equation that expresses an equivalence between an expression that is strictly imaginary and another expression that is strictly Real*. How is that possible? Real numbers and imaginary numbers are completely distinct from one another, they represent two entirely different and *mutually exclusive* number sets. Also, recall that the very concept of the imaginary number is artificial – it's a *man-made notion* devised to facilitate solving equations. Yet here we have an expression solely involving imaginary numbers (i.e., the left-side) *that is equivalent to* an expression solely involving Real values (the right-side). How can an imaginary quantity be equal to a Real quantity? Again, by their very definitions they are supposed to be mutually exclusive. The very existence of such an equation would seem like a mathematical impossibility. And to add intrigue to incredulity, we have the additional mind-blower that the real side of the equation involves those two extremely important constants ***π*** and ***e*.** How is it that, of all the infinitude of real numbers out there, *those two* real numbers are the ones that come into play here? (As we will shortly see, there is another property of the numbers ***π*** and ***e*** that leads to an even more astonishing fact, a fact that I suspect may be the most important mathematical concept of all).

It's all too easy to look upon something like this equation and simply shrug one's shoulders, scratch one's head, and say, "Huh! That's odd." But the fact is that something this strange and impossible is usually a beacon pointing the way to a deeper insight of profound import. There is a very good reason why author Isaac Asimov said, "The most exciting phrase to hear in science, the one that heralds new discoveries, is not 'Eureka!' (I found it!) but 'Gee, that's funny ...' " It is often those *"Gee, that's funny ..."* situations that point the way to gold. As such, they demand to be explored.

Well... How's that for taking stuffy old math and breathing a little excitement and mystery into it? Now that we've talked about imaginary numbers, and, in the process, laid the groundwork regarding sets of numbers, let's move on to another topic of mathematics that is just as strange and implausible and intriguing as imaginary numbers.

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{*This chapter continues in the book*}

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[i] "Set (mathematics)" *Wikipedia: The Free Encyclopedia*. Wikimedia Foundation, Inc. 22 July 2004. Web. 6 Mar. 2014. Accessed 8 Mar. 2014, <http://en.wikipedia.org/wiki/Set_%28mathematics%29>.

[ii] James R. Newman, The World of Mathematics, Volume One (Redmond, Washington: Tempus Books of Microsoft Press, 1956, 1988) pp 28-29.

[iii] *op. cit*., Pickover, p. 124; "Applications of Imaginary Numbers" *The Math forum@Drexel: Ask Dr. Math*. 14 Oct. 1997. Accessed 10 Mar. 2014. <http://mathforum.org/library/drmath/view/ 53606.html>; "Using Imaginary Numbers" *The Math forum@Drexel: Ask Dr. Math*. 4 May 2001. Accessed 10 Mar. 2014. < http://mathforum.org/library/drmath/view/53879.html>.

[iv] *op. cit*., Newman, p. 29.

[v] "A Visual, Intuitive Guide to Imaginary Numbers" *Better Explained*. Web. 21 Dec. 2007. Accessed 10 Mar. 2014. < http://betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/>.

[vi] Ibid.